

The Amazing Image Conjecture

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Abstract

In this paper we discuss a general framework in which we present a new conjecture, due to Wenhua Zhao, the Image Conjecture. This conjecture implies the Generalized Vanishing Conjecture and hence the Jacobian Conjecture. Crucial ingredient is the notion of a Mathieu space: let k be a field and R a k -algebra. A k -linear subspace M of R is called a Mathieu subspace of R , if the following holds: let $f \in R$ be such that $f^m \in M$, for all $m \geq 1$, then for every $g \in R$ also $gf^m \in M$, for almost all m , i.e. only finitely many exceptions.

Let A be the polynomial ring in $\zeta = \zeta_1, \dots, \zeta_n$ and z_1, \dots, z_n over \mathbb{C} . The Image Conjecture (IC) asserts that $\sum_i (\partial_{z_i} - \zeta_i)A$ is a Mathieu subspace of A . We prove this conjecture for $n = 1$. Also we relate (IC) to the following Integral Conjecture: if B is an open subset of \mathbb{R}^n and σ a positive measure, such that the integral over B of each polynomial in z over \mathbb{C} is finite, then the set of polynomials, whose integral over B is zero, is a Mathieu subspace of $\mathbb{C}[z]$. It turns out that Laguerre polynomials play a special role in the study of the Jacobian Conjecture. ¹

Introduction

Some twenty five years ago I learnt about the existence of the Jacobian Conjecture, during one of my visits to my friend Pascal Adjamagbo in Paris. The problem always stayed somewhere in my mind and in the meantime I worked

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on different related topics and found counter examples to various conjectures, which would have implied the truth of the still mysterious Jacobian Conjecture. All these experiences fed my believe that the Jacobian Conjecture, if true at all, would be difficult to generalize, since it felt like a kind of optimal statement. Therefore I often stated in public the following dictum

“If you have a conjecture which implies the Jacobian Conjecture, but is not equivalent to it, then you can be sure that your conjecture is false.”

It is therefore no surprise that, when in July 2009 Wenhua Zhao came up with a new conjecture implying the Jacobian Conjecture, I set out to find a counter example. This conjecture, which was given the name *Image Conjecture* by its inventor, is so general that I was convinced that it would be easy to find a counterexample. Surprisingly, I did not. Instead I found various instances in favour of it.

The aim of this paper is to bring this fascinating new conjecture to the attention of a larger audience. Hopefully it will inspire the reader to join me in my search for either a proof, or counterexample. The style in which it is written will be easy going. Sometimes I will skip proofs and refer to the papers of Zhao and the upcoming joint work with Wright and Zhao ([EWZ]).

1 The Image Conjecture: a first encounter

To please those readers who cannot wait to see what the Image Conjecture is all about, I will start this section, by describing its most important special case.

Let k be any field, A a commutative k -algebra and $A[z]$ the polynomial ring in n variables $z = (z_1, \dots, z_n)$ over A . Elements of the ring $A[z]$ will simply be called polynomials, without referring to A or z . Let a_1, \dots, a_n be elements of A and denote by D the set of commuting differential operators

$$\partial_{z_1} - a_1, \dots, \partial_{z_n} - a_n.$$

Finally put

$$ImD = \sum_{i=1}^n (\partial_{z_i} - a_i)A[z].$$

Image Conjecture (IC(n,A)). Assume that (a_1, \dots, a_n) is a regular sequence in A . If all positive powers of a polynomial f belong to ImD , then for every polynomial g , almost all polynomials gf^m also belong to ImD .

The sentence *almost all* means for all, with only a finite number of exceptions. Furthermore the sequence (a_1, \dots, a_n) is called a *regular sequence in A* if a_1 is no zero-divisor in A , for each $i \geq 1$ the element a_{i+1} is no zero-divisor in $A/(a_1, \dots, a_i)$ and the ideal generated by all a_i is not equal to A .

To get a feeling for the difficulty of the problem, the reader is invited to find a proof for the one dimensional case. In fact, in this dimension the conjecture has only been proved in case the ideal Aa_1 is a radical ideal. If additionally A is a UFD, also the non-radical case has been proved. As we will see below, the Jacobian Conjecture follows from the very special case where A is the polynomial ring $\mathbb{C}[\zeta_1, \dots, \zeta_n]$ and $a_i = \zeta_i$ for each i .

The property concerning the powers of polynomials, which is used in the formulation of the Image Conjecture above, was formalized by Zhao in his paper [Z4] as follows.

Let k be a field, R a k -algebra (not necessarily commutative) and M a k -linear subspace of R . An element f of R is said to have the *left Mathieu property with respect to M* , if the following holds: if all positive powers of f belong to M , then for every g of R , almost all elements gf^m belong to M . Furthermore M is called a *left Mathieu subspace of R* , if all elements of R have the left Mathieu property, with respect to M . Similarly one defines the notion of a right Mathieu subspace and finally M is called a Mathieu subspace, if it is both a left and right Mathieu subspace. In most examples discussed in this paper, the ring R will be commutative, so we just speak about Mathieu subspaces.

Using this terminology, the Image Conjecture formulated above simply states that ImD is a Mathieu subspace of $A[z]$.

The notion of Mathieu subspace was first introduced by Zhao in [Z4] and was inspired by the following conjecture proposed by Olivier Mathieu in [M], 1995.

Mathieu Conjecture. *Let G be a compact connected real Lie group with Haar measure σ . Let f be a complex valued G -finite function on G , such that $\int_G f^m d\sigma = 0$ for all positive m . Then for every G -finite function g on G , also $\int_G gf^m d\sigma = 0$, for almost all m .*

Here a function f is called *G -finite*, if the \mathbb{C} -vector space generated by the elements of the orbit $G \cdot f$ is finite dimensional.

With the terminology introduced above, the Mathieu Conjecture can be reformulated as follows: let R be the \mathbb{C} -algebra of complex valued G -finite

functions on G . Then the \mathbb{C} -subspace of f 's, which satisfy $\int_G f d\sigma = 0$, is a Mathieu subspace of R .

The importance of Mathieu's conjecture comes from the fact that it implies the Jacobian conjecture, as was shown in [M]. Since its formulation, only one non-trivial case of this conjecture was solved, namely the case that G is commutative. This result, which is due to Duistermaat and van der Kallen, can be formulated as follows (see [DvK]).

Duistermaat-van der Kallen theorem. *Let k be a field of characteristic zero and $R = k[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}]$, the ring of Laurent polynomials over k . Then the set of Laurent polynomials, whose constant term is zero, is a Mathieu subspace of R .*

Already the proof of the one dimensional case is not at all obvious and again the reader is invited to find an elementary proof. The hypothesis that k has characteristic zero, cannot be dropped, as can be seen from the following example, which is due to Roel Willems.

Counterexample 1.1. *Let $n = 1$ and write t instead of z_1 . Let $f = t^{-1} + t^{p-1} \in k[t, t^{-1}]$, where k is a field of characteristic $p > 0$. Then the constant term of all positive powers of f is zero, however for all $m = p^k - 1$, the constant coefficient of $t^{-1}f^m$ is non-zero.*

The notion of Mathieu subspaces of a ring R can be viewed as a *generalization of ideals rings*, since obviously an ideal of R is a Mathieu subspace of R . However, Mathieu subspaces are far more complicated to understand and to recognize. For example it is easy to describe all ideals of the univariate polynomial ring $k[t]$, but even for concrete cases we have no way, other than ad hoc methods, to decide if a given k -linear subspace of $k[t]$ is a Mathieu subspace or not. The reader who wants to test this statement is referred to section six, where we discuss various Mathieu subspaces of $k[t]$ and some candidate ones.

There is however one easy property that Mathieu subspaces share with ideals and which often can be used to show that a given subspace is not a Mathieu space.

The 1-property. *Let M be a Mathieu subspace of a k -algebra R . If M contains 1, then $M = R$.*

Indeed, if 1 belongs to M , then all positive powers of 1 belong to M . Hence, by the Mathieu property, it follows that for each g in R almost all elements $g \cdot 1^m$ belong to M , i.e. each such g belongs to M . So $M = R$.

Example 1.2. Let $R = k[z]$ be the univariate polynomial ring and k a field of positive characteristic p . Let $D = \partial_z$. Then $\text{Im}D$ is not a Mathieu subspace of R .

Namely obviously $\text{Im}D$ contains 1, but it does not contain z^{p-1} since $p = 0$ in R . So by the 1-property $\text{Im}D$ cannot be a Mathieu subspace of R .

This example shows that the hypothesis concerning the regularity of the sequence of a_i 's in the statement of the Image Conjecture cannot be dropped.

2 Motivation for the Image Conjecture

To understand where the Image Conjecture comes from, we recall some recent results concerning the Jacobian Conjecture.

As is well-known the Jacobian Conjecture was formulated by O. Keller in 1939 in his paper [K]. It asserts that if the jacobian determinant of a polynomial map from complex affine n -space to itself is a non-zero constant, then the map is invertible, in the sense that its inverse is again a polynomial map. The conjecture is open in all dimensions n greater than one. In 1982, Bass, Connell and Wright, and independently Yagzhev showed that in order to prove or disprove the conjecture it suffices to study so-called *cubic homogeneous* polynomial maps i.e. maps of the form

$$z + H = (z_1 + H_1, \dots, z_n + H_n)$$

where the H_i are either zero or homogeneous of degree three. This result is known as the *cubic homogeneous reduction*. It was also shown that the condition for the jacobian determinant of such a map to be a non-zero constant is equivalent to the nilpotency of the jacobian matrix of H (see [BCW] or [E]).

In 2003 Michiel de Bondt and the author improved upon the above reduction result, by showing that one may *additionally assume* that the Jacobian matrix of H is symmetric (see [BE]), which by Poincaré's lemma implies that H equals the gradient of some (quartic) polynomial P in n variables over \mathbb{C} i.e. $P \in \mathbb{C}[z]$. Using this fact Wenhua Zhao obtained the following new and surprising *equivalent* description of the Jacobian Conjecture ([Z1], 2004)

Vanishing Conjecture. Let $\Delta = \sum_i \partial_{z_i}^2$ be the Laplace operator and let $P \in \mathbb{C}[z]$ be homogeneous. If $\Delta^m(P^m) = 0$ for all positive m , then $\Delta^m(P^{m+1}) = 0$ for almost all m .

In fact the condition $\Delta^m(P^m) = 0$ for all positive m is equivalent to the nilpotency of the jacobian matrix $J(\nabla(P))$ and the vanishing of all sufficiently large powers $\Delta^m(P^{m+1})$ is equivalent to the invertibility of the map $z + \nabla(P)$.

Zhao observed the resemblance with Mathieu's conjecture and could make this resemblance even better by showing that the Vanishing Conjecture is equivalent to the following version (see [EZ])

Vanishing Conjecture. *If $P \in \mathbb{C}[z]$ is homogeneous and such that $\Delta^m(P^m) = 0$ for all positive m , then for each Q in $\mathbb{C}[z]$ $\Delta^m(QP^m) = 0$ for almost all m .*

Now it is not difficult to show that if the Vanishing Conjecture holds for the Laplace operator, it also holds for all quadratic homogeneous operators with constant coefficients (use Lefschetz's principle and the fact that over the complex numbers all quadratic forms are essentially sums of squares. For more details we refer to [Z2]).

After these observations Zhao made the following more general conjecture, dropping the homogeneity condition on the polynomial P and replacing the Laplace operator by any differential operator with *constant coefficients*

Generalized Vanishing Conjecture(GVC(n)). *Let Λ be any differential operator with constant coefficients, i.e. $\Lambda \in \mathbb{C}[\partial_1, \dots, \partial_n]$. If $P \in \mathbb{C}[z]$ is such that $\Lambda^m(P^m) = 0$ for all positive m , then for each Q in $\mathbb{C}[z]$ also $\Lambda^m(QP^m) = 0$ for almost all m .*

When I saw this conjecture I was convinced that it should be easy to find a counterexample. So I first investigated the one dimensional case. Then the conjecture is easily seen to be true, namely let Λ be a polynomial in $\mathbb{C}[\partial]$ of order $r \geq 0$, i.e. ∂^r is the lowest degree monomial in ∂ appearing in Λ . Now let P in $\mathbb{C}[z]$ be a polynomial of degree d . Observe that the polynomial $\Lambda(P)$ has degree $d - r$ if $r \leq d$. In particular, if $\Lambda(P) = 0$ it follows that $r \geq d + 1$. Consequently the order of Λ^m , which equals rm , is greater or equal to $dm + m$, which is greater than the degree of QP^m if m is greater than the degree of Q . This implies that $\Lambda^m(QP^m) = 0$ for such m .

Next I investigated the special two variable case $\partial_1^p + \partial_2^q$ where p and q are natural numbers with greatest common divisor 1. Also in this case the conjecture turned out to be true. Many more special cases have been proved since. The reader is referred to the paper [EWZ], where several of them have been established.

Studying the Generalized Vanishing Conjecture, Zhao observed that the

main obstruction to understand the condition $\Lambda^m(P^m) = 0$ is the fact that the differential operator Λ and the multiplication operator P do not commute. Therefore he considered the *left symbol map* \mathfrak{L} from the Weyl algebra $A_n(\mathbb{C})$ to the polynomial ring $\mathbb{C}[\zeta, z]$, which is the \mathbb{C} -linear map sending each monomial $\partial^a z^b$ to $\zeta^a z^b$. So to compute the image of a differential operator under this map, one first needs to write the operator as a \mathbb{C} -linear combination of monomials of the form $\partial^a z^b$ and then replace each ∂_i by ζ_i . In a similar way one can define the *right symbol map* \mathfrak{R} from the Weyl algebra to the polynomial ring $\mathbb{C}[\zeta, z]$ by defining $\mathfrak{R}(z^a \partial^b) = z^a \zeta^b$.

Now let \circ denote the multiplication in the Weyl algebra $A_n(\mathbb{C})$, i.e. the composition as \mathbb{C} -linear maps acting on the polynomial ring $\mathbb{C}[z]$. Then the condition $\Lambda^m(P^m) = 0$ is equivalent to $(\Lambda^m \circ P^m)(1) = 0$. Furthermore, the differential operators which annihilate the element 1 form the left ideal in $A_n(\mathbb{C})$ generated by the partial derivatives ∂_i . Applying the right symbol map \mathfrak{R} , we therefore obtain that $\Lambda^m(P^m) = 0$, if and only if $\mathfrak{R}(\Lambda^m \circ P^m)$ belongs to the ideal generated by the ζ_i in $\mathbb{C}[\zeta, z]$, or equivalently that $\pi \circ \mathfrak{R}(\Lambda^m \circ P^m) = 0$, where π denotes the $\mathbb{C}[z]$ -homomorphism from $\mathbb{C}[\zeta, z]$ to $\mathbb{C}[z]$ sending each ζ_i to zero. Finally observe that

$$\mathfrak{L}(\Lambda^m \circ P^m) = \Lambda(\zeta)^m P(z)^m.$$

Since \mathfrak{L} is an isomorphism of \mathbb{C} -vector spaces this implies that

$$\Lambda^m \circ P^m = \mathfrak{L}^{-1}(\Lambda(\zeta)^m P(z)^m).$$

Combining this with the observation above, we obtain that

$$\Lambda^m(P^m) = 0 \text{ if and only if } L((\Lambda(\zeta)P(z))^m) = 0,$$

where $L = \pi \circ \mathfrak{R} \circ \mathfrak{L}^{-1}$. In a similar way the condition $\Lambda(QP^m) = 0$ is equivalent to $L(Q(z)(\Lambda(\zeta)P(z))^m) = 0$. So combining both results above the Generalized Vanishing Conjecture can be reformulated as follows: let $f = \Lambda(\zeta)P(z)$ and $g = Q(z)$. If f^m belongs to $\ker L$ for all positive m , then gf^m belongs to $\ker L$ for almost all m . Having Mathieu's observations in mind it then was a minor step to generalize this conjecture to the stronger statement that the above implication should hold for all f and g in $\mathbb{C}[\zeta, z]$. In other words these calculations led Zhao to conjecture that $\ker L$ is a Mathieu subspace of $\mathbb{C}[\zeta, z]$.

Then the next natural question to consider is: is there a nice way to describe $\ker L$? To answer this question, let's look at the definition of L .

It is an easy exercise to verify that $L(\zeta^a z^b) = \partial^a(z^b)$. In other words the image of a polynomial in ζ and z under L is obtained as follows: write the polynomial as a \mathbb{C} -linear combination of monomials of the form $\zeta^a z^b$, then replace ζ by ∂ and apply ∂ to the monomial in z . From this description, one deduces readily that for each polynomial g in $\mathbb{C}[\zeta, z]$, the element of $\partial_{z_i}(g) - \zeta_i g$ belongs to $\ker L$ and more generally that $\text{Im} D \subset \ker L$, where D is the set of n commuting operators $\partial_{z_i} - \zeta_i$ and $\text{Im} D$ is as defined in section one. Then finally Zhao showed that in fact $\text{Im} D$ is equal to $\ker L$, which by the above arguments led to his original formulation of the Image Conjecture. A proof of the equality of $\ker L$ and $\text{Im} D$ is given in the next section, where we also show that, as indicated by the arguments above, the Image Conjecture (for the ring $A = \mathbb{C}[\zeta_1, \dots, \zeta_n]$) implies the Generalized Vanishing Conjecture.

3 The Image Conjecture and the Generalized Vanishing Conjecture

In this section we show that the Image Conjecture implies the Generalized Vanishing Conjecture (and hence the Jacobian Conjecture). More precisely, we only need the following special case of the Image Conjecture: $A = \mathbb{C}[\zeta] = \mathbb{C}[\zeta_1, \dots, \zeta_n]$ and $a_i = \zeta_i$ for each i . Let's denote this case by $IC(n)$ for simplicity.

Theorem 3.1. *$IC(n)$ implies $GVC(n)$.*

Let L be the \mathbb{C} -linear map from $\mathbb{C}[\zeta, z]$ to $\mathbb{C}[z]$ defined in the previous section. In other words

$$L(\zeta^a z^b) = \partial_z^a(z^b).$$

We will show below that we have the equality

$$\text{Im} D = \ker L.$$

Before we prove this result let us first show how it can be used to prove the theorem. So let $\Lambda = \Lambda(\partial)$ be a differential operator with constant coefficients and P a polynomial in $\mathbb{C}[z]$ such that $\Lambda^m(P^m) = 0$ for all positive m . Let $g \in \mathbb{C}[z]$. We must show that also $\Lambda^m(gP^m) = 0$ for almost all m . Therefore put

$$f(\zeta, z) = \Lambda(\zeta)P(z).$$

It then follows from the hypothesis and the definition of L that $L(f^m) = 0$ for all positive m . Using the equality $\ker L = \text{Im} D$, this implies that all positive powers of f belong to $\text{Im} D$, which by assumption is a Mathieu subspace of $\mathbb{C}[\zeta, z]$. Consequently also gf^m belongs to $\text{Im} D = \ker L$ for almost all m . So $L(gf^m) = 0$ for almost all m . Using the definition of f this gives that $L(\Lambda(\zeta)^m g(z)P(z)^m) = 0$, whence $\Lambda^m(gP^m) = 0$ for almost all m . So $GVC(n)$ holds.

To conclude this section we prove the equality $\text{Im} D = \ker L$. For another proof we refer the reader to [Z3]. Our proof is based on the following result from the theory of \mathfrak{D} -modules

Proposition 3.2. *Let M be an $A_n(\mathbb{C}) = \mathbb{C}[t_1, \dots, t_n, \partial_1, \dots, \partial_n]$ -module such that each ∂_i is locally nilpotent on M . Then each f in M can be written uniquely in the form $f = \sum t^a f_a$, where each f_a belongs to $N = \bigcap_i \ker(\partial_i, M)$. In particular $f \in \sum t_i M$ if and only if $f_0 = 0$.*

To obtain the equality $\text{Im} D = \ker L$ we apply this proposition to $M = \mathbb{C}[\zeta, z]$, which is an $A_n(\mathbb{C})$ -module by defining

$$t_i f = (\zeta_i - \partial_{z_i})f \text{ and } \partial_i f = \partial_{\zeta_i}(f).$$

Then $N = \bigcap \ker \partial_{\zeta_i} = \mathbb{C}[z]$. So $f \in \mathbb{C}[\zeta, z]$ can be written uniquely in the form

$$f = \sum (\zeta_i - \partial_{z_i})^a f_a(z) \quad (*)$$

for some $f_a(z)$ in $\mathbb{C}[z]$, and in particular $f \in \sum (\partial_{z_i} - \zeta_i)\mathbb{C}[\zeta, z] = \text{Im} D$ if $f_0(z) = 0$. Finally observe that $(*)$ implies that $L(f) = f_0(z)$. So we get that f belongs to $\text{Im} D$ if f belongs to $\ker L$. So $\ker L \subset \text{Im} D$. Since we saw that $\text{Im} D \subset \ker L$ in the previous section, the desired equality follows.

The proof of the proposition above follows easily by applying the next lemma repeatedly

Lemma 3.3. *Assume that M is an $A_1(\mathbb{C}) = \mathbb{C}[t, \partial]$ -module and f an element of M such that $\partial^m f = 0$ for some m . Then $f = f_0 + t f_1 + \dots + t^{m-1} f_{m-1}$ for some $f_i \in N = \ker \partial$ which are uniquely determined.*

Proof. The uniqueness follows easily by applying ∂ sufficiently many times. So let $\partial^m f = 0$. Then $\partial^{m-1}(\partial f) = 0$, so by induction on m we get

$$\partial f = g_0 + tg_1 + \dots + t^{m-2}g_{m-2}$$

for some g_i in N . Now let

$$G := \int \partial f = tg_0 + \frac{1}{2}t^2g_1 + \dots + \frac{1}{m-1}t^{m-1}g_{m-2}.$$

Then $\partial G = \partial f$, so $f - G \in N$. Say $f - G = f_0$ for some f_0 in N . Using the definition of G the desired result follows.

4 A surprising connection

In the mean time Zhao had taken a completely different approach. He just wondered if sequences of the form $\Lambda P, \Lambda^2(P^2), \Lambda^3(P^3), \dots$, where P can be any polynomial, had been studied before. At the same time he took a closer look at the Laplace operator by compactifying real n -space, which led him to investigate eigen functions of this operator on the n -sphere. Studying the literature he came in contact with the Gegenbauer polynomials, a special class of orthogonal polynomials. In particular he found the classical Rodrigues' formula, which gives a useful way to describe these polynomials. Before I continue this story, let me first recall some basic facts concerning orthogonal polynomials (see also [DX] and [S]).

Orthogonal polynomials

Let B be a non-empty open subset of \mathbb{R}^n and w a so-called *weight function* on B i.e., it is strictly positive on B and its integral over this set is finite and positive. To such a function one can associate a *Hermitian inner product* on the n -dimensional polynomial ring $\mathbb{C}[x]$ by defining

$$\langle f, g \rangle = \int_B f(x) \overline{g(x)} w(x) dx.$$

A set of polynomials u_a , where $a = (a_1, \dots, a_n)$ runs through \mathbb{N}^n , is called *orthogonal over B* with respect to the weight function w , if they form an orthogonal basis of $\mathbb{C}[x]$ with respect to the associated inner product described

above and satisfy the additional condition that the degree of each polynomial u_a is equal to $|a|$, the sum of all a_i .

A standard way to construct orthogonal polynomials in one variable is to apply the Gram-Schmidt process to the basis $1, x, x^2, \dots$. Making special choices for B and w gives the following so-called *classical orthogonal polynomials*.

1. The *Hermite polynomials*: $B = \mathbb{R}$ and $w(x) = e^{-x^2}$.
2. The *Laguerre polynomials*: $B = (0, \infty)$ and $w(x) = x^\alpha e^{-x}$, with $\alpha > -1$.
3. The *Jacobi polynomials*: $B = (-1, 1)$ and $w(x) = (1-x)^\alpha(1+x)^\beta$, with $\alpha, \beta > -1$. In case both parameters are zero, i.e. $w = 1$, the polynomials are called *Legendre polynomials*.

From univariate orthogonal polynomials one can construct orthogonal polynomials in dimension n as follows: for each $1 \leq i \leq n$ choose an open subset B_i of \mathbb{R} and a weight function w_i on B_i . Let $u_{i,m}$ with $m \geq 0$ be an orthogonal set of univariate polynomials with respect to B_i and w_i .

Then $B = B_1 \times \dots \times B_n$ is an open subset of \mathbb{R}^n and w defined by $w(x) = w_1(x_1) \dots w_n(x_n)$ is a weight function on B , where $x = (x_1, \dots, x_n)$. Furthermore one easily verifies that the polynomials

$$u_a(x) = u_{1,a_1}(x_1), \dots, u_{n,a_n}(x_n)$$

where $a = (a_1, \dots, a_n)$, form an orthogonal set of polynomials over B with respect to the weight function w .

The multivariate orthogonal polynomials obtained from Hermite polynomials will again be called Hermite polynomials. Similarly we get multivariate Laguerre and Jacobi polynomials. These polynomials we call the *classical (multivariate) orthogonal polynomials*.

Now a surprising result is that all these classical orthogonal polynomials can be obtained from the so-called Rodrigues' formula. With the terminology introduced above it asserts the following

Rodrigues' formula. *Let u_a be a system of classical orthogonal polynomials. Then there exist a $g = (g_1, \dots, g_n)$ in $\mathbb{C}[x]^n$ and non-zero real constants c_a such that*

$$u_a = c_a w^{-1} \partial_x^{|a|} (w g^a)$$

For example in the one-dimensional case, one obtains the Hermite, Laguerre and Jacobi polynomials by taking $g = 1, x, 1 - x^2$ respectively and the constants c_a are respectively equal to $(-1)^a$, $\frac{1}{a!}$ and $\frac{(-1)^a}{2^a a!}$.

Then Zhao made a remarkable discovery, namely if one defines

$$\Lambda_i = w^{-1} \circ \partial_i \circ w = \partial_i + w^{-1} \partial_i(w)$$

and

$$\Lambda = (\Lambda_1, \dots, \Lambda_n)$$

then

$$u_a = c_a \Lambda^a(g^a) \quad (1)$$

In other words all classical orthogonal polynomials come from sequences of the form $\Lambda^a(g^a)$. For example in the one-dimensional case, taking for Λ the operators $\partial - 2x$, $\partial + (\alpha x^{-1} - 1)$ and $\partial - \alpha(1 - x)^{-1} + \beta(1 + x)^{-1}$, one gets the Hermitian, Laguerre and the Jacobi polynomials respectively, apart from the constants described above.

Zhao was struck by the fact that, just as in the formulation of the Image Conjecture, again differential operators of order one appeared. He therefore wondered if a similar kind of Image Conjecture would hold for the commuting set of differential operators Λ coming from the classical orthogonal polynomials as described above. He made the following modified conjecture

Image Conjecture for Classical Orthogonal Polynomials.

Let Λ be as described above and let

$$Im' \Lambda := \mathbb{C}[z] \cap \left(\sum_i \Lambda_i(\mathbb{C}[z]) \right) \quad (2)$$

Then $Im' \Lambda$ is a Mathieu subspace of $\mathbb{C}[z]$.

Now let's take a closer look at the intersection described in (2). Using the notation for g and u_a introduced above, one can verify by explicit calculation that in the univariate case $\Lambda^m(g^a)$ is a polynomial, if m is at most a . From this and (1) one obtains that each multivariate classical orthogonal polynomial u_a , where a is not the zero vector, belongs to $Im' \Lambda$. These observations lead to the following interesting result.

Proposition 4.1. *Notations as above. Assume that 1 does not belong to $Im'\Lambda$. Then*

- i) $Im'\Lambda$ is the \mathbb{C} -linear span of all u_a where a is non zero.*
- ii) $Im'\Lambda = \{f \in \mathbb{C}[x] \mid \int_B f w dx = 0\}$.*

Proof. i) follows from (2), the hypothesis and the observation above proposition 4.1.

ii) Let $f \in \mathbb{C}[x]$. Write it in the basis $\{u_a\}$, say $f = \sum f_a u_a$ with f_a in \mathbb{C} . Since $u_0 \in \mathbb{C}^*$, it follows from i) that f belongs to $Im'\Lambda$ if and only if $f_0 = 0$. Since the $\{u_a\}$ form an orthogonal basis and $u_0 \in \mathbb{C}^*$ we have that

$$f_0 \langle u_0, u_0 \rangle = \langle f, u_0 \rangle = \int_B f \cdot \overline{u_0} \cdot w dx = \overline{u_0} \int_B f w dx.$$

So f belongs to $Im'\Lambda$ if and only if $\int_B f w dx = 0$.

Inspired by Mathieu's conjecture, replacing the connected compact Lie group G by the open subset B of \mathbb{R}^n , the Haar measure by a positive measure $d\sigma$ and the G -finite functions by polynomials, the above proposition led Zhao to the following analogue of Mathieu's conjecture

Zhao's Integral Conjecture. *Let B be an open subset of \mathbb{R}^n and σ a positive measure on B such that for any polynomial g in $\mathbb{C}[x]$ the integral $\int_B g d\sigma$ is finite. Then the set of polynomials f whose integral over B is zero, is a Mathieu subspace of $\mathbb{C}[x]$.*

This conjecture is widely open in all dimensions. We will return to it in section 6 below. For now we show

Corollary 4.2. *The Integral Conjecture implies the Image Conjecture for classical orthogonal polynomials.*

Proof. Since a weight function w is strictly positive on B , the measure $d\sigma = w dx$ is positive on B . Consequently if 1 does not belong to $Im'\Lambda$, the above proposition together with the Integral Conjecture imply that this set is a Mathieu subspace of $\mathbb{C}[x]$. In case 1 does belong to $Im'\Lambda$, the observation immediately before proposition 4.1 implies that this image is the whole polynomial ring and hence it is a Mathieu subspace as well.

The Image Conjecture and the Image Conjecture for classical orthogonal polynomials inspired Zhao in [Z3] to formulate the following rather general statement

General Image Conjecture. *Let k be a field, A a commutative k -algebra and $A[z] = A[z_1, \dots, z_n]$ the polynomial ring over A . Let D be a commuting set of differential operators of the form*

$$\sum_{i=1}^n c_i \partial_i + g(z)$$

where the c_i belong to A and $g(z) \in A[z]$. Then

$$\text{Im} D := \sum_{\Lambda \in D} \Lambda A[z]$$

is a Mathieu subspace of $A[z]$.

Comment 1. If the field k has characteristic zero, no counterexamples to this conjecture are known. On the other hand, as we have seen in the example at the end of section one, the conjecture is false if the characteristic of k is positive. At this moment it is not clear what extra condition, similar to the one given in the statement of the Image Conjecture, can be added to avoid this kind of obvious counterexamples.

Comment 2. It is shown in [Z3] that if the field k has characteristic zero, then in order to prove or disprove the General Image Conjecture for sets of operators for which the c_i belong to k (not just in A), it suffices to study the cases where the set D consists of n operators of the form $\partial_i - \partial_i(q(z))$, where $q(z)$ is some polynomial. Observe that if we take for $q(z)$ the linear form $a_1 z_1 + \dots + a_n z_n$ we obtain the statement of the Image Conjecture as described in section one.

After having described various conjectures, it is time to investigate the question: what evidence is there in favour of these conjectures and in particular what evidence supports the Image Conjecture?

5 The Image Conjecture in positive characteristic

Throughout this section (except in the crucial lemma below) k will be a field of characteristic $p > 0$. With the notations introduced in section one we get

Theorem 5.1 *If (a_1, \dots, a_n) is a regular sequence in A , then $\text{Im}D$ is a Mathieu subspace of $A[z]$. In other words the Image Conjecture is true for positive characteristic.*

The proof of this theorem is based on the following result, whose proof will be sketched at the end of this section (we refer to [EWZ] for more details).

Crucial lemma. *Let k be any field. Let b be a polynomial of degree d and denote by b_d its homogeneous component of degree d . If b belongs to $\text{Im}D$, then all coefficients of b_d belong to the ideal I of A generated by all a_i .*

Corollary 5.2. *Let f be a sum of monomials $f_a z^a$. If f^p belongs to $\text{Im}D$ then each f_a^p belongs to I .*

Proof. Write f as a sum of homogeneous components f_i . Then f^p is a sum of f_i^p . It then follows from the crucial lemma that all coefficients of f_d^p belong to I , where d is the degree of f . So f_d^p is a sum of monomials of the form $c a_i z^{ap}$, with $|a| = d$. Since each such a monomial is equal to $(\partial_i - a_i)(-c z^{ap})$, which belongs to $\text{Im}D$, it follows that f_d^p belongs to $\text{Im}D$. Subtracting this polynomial from f^p we obtain that

$$f_0^p + \dots + f_{d-1}^p \in \text{Im}D.$$

Then the result follows by induction on d .

Proof of the theorem 5.1.

i) Since $\partial^p = 0$ on $A[z]$, we get that $a_i^p z^a = (\partial_i - a_i)^p(-z^a) \in \text{Im}D$. So a polynomial belongs to $\text{Im}D$ if all its coefficients belong to the ideal J generated by all a_i^p .

ii) Now let f be such that f^p belongs to $\text{Im}D$ and g be any polynomial. By i) it suffices to show that all coefficients of gf^m belong to J if $m \geq p^2$. Therefore write f as a sum of monomials $f_a z^a$. Since f^p belongs to $\text{Im}D$, it follows from corollary 5.2 that each f_a^p belongs to I and hence each $f_a^{p^2}$ belongs to J . Since f^{p^2} is a sum of the monomials $f_a^{p^2} z^{p^2 a}$, it then readily follows that all coefficients of gf^m belong to J if $m \geq p^2$. As observed above this concludes the proof.

Proof of crucial lemma (sketch).

We only sketch the case $n = 2$. Let $b \in \text{Im}D$, say

$$b = (\partial_1 - a_1)p + (\partial_2 - a_2)q \quad (*)$$

for some polynomials p and q . Let d be the degree of b and denote by b_d the homogeneous component of degree d . Now we assume for simplicity that the degrees of both p and q are at most $d + 2$. Then looking at the component of degree $d + 2$ in (*) we get

$$-a_1 p_{d+2} - a_2 q_{d+2} = 0$$

where p_i and q_i denote the homogeneous components of degree i of p and q respectively. From the regularity hypothesis on the sequence a_1, a_2 it then follows that there exists a polynomial g_{d+2} , homogeneous of degree $d + 2$, such that

$$p_{d+2} = a_2 g_{d+2} \text{ and } q_{d+2} = -a_1 g_{d+2}.$$

Comparing the components of degree $d + 1$ in the equation (*) we get

$$\partial_1 p_{d+2} + \partial_2 q_{d+2} - a_1 p_{d+1} - a_2 q_{d+1} = 0.$$

Substituting the formulas for p_{d+2} and q_{d+2} , found above, gives

$$a_1(p_{d+1} + \partial_2 g_{d+2}) - a_2(q_{d+1} - \partial_1 g_{d+2}) = 0.$$

Again, from the regularity condition it then follows that there exists some polynomial g_{d+1} , homogeneous of degree $d + 1$, such that

$$p_{d+1} = -\partial_2 g_{d+2} + a_2 g_{d+1} \text{ and } q_{d+1} = \partial_1 g_{d+2} + a_1 g_{d+1}.$$

Comparing the components of degree d in the equation (*) we get

$$b_d = \partial_1 p_{d+1} + \partial_2 q_{d+1} - a_1 p_d - a_2 q_d.$$

Finally, substituting the formulas for p_{d+1} and q_{d+1} in the last equality gives

$$b_d = -a_1(p_d - \partial_2 g_{d+1}) - a_2(q_d - \partial_1 g_{d+1}).$$

which gives the desired result.

6 Examples of Mathieu subspaces

Before we give some more evidence in favour of the Image Conjecture we want to discuss various examples of Mathieu spaces. The first example concerns the ring of $n \times n$ matrices over a field k , where either the characteristic of k is zero or greater than n .

Example 6.1. *Let R be the ring of $n \times n$ matrices over k . Then the subspace consisting of all matrices having trace zero is a Mathieu subspace of R .*

Indeed, it is well-known that if the traces of the first n powers of a matrix A are zero, then the matrix is nilpotent and hence its n -th power is the zero matrix. Consequently for any matrix B also $BA^m = 0$ if m is at least n . In particular the trace of this matrix is zero for all $m \geq n$.

Remark 6.2. *It is proved in [Z5] that, under the hypothesis on the characteristic of k described above, the subspace of Example 6.1 is the only co-dimension one left or right Mathieu subspace of R . In case the characteristic is positive and at most n , it turns out that R has no left or right Mathieu subspaces of co-dimension one.*

Now let me give two less trivial examples. Both concern subspaces of the univariate polynomial ring $\mathbb{C}[t]$. Again the reader is invited to find elementary proofs for the following two results.

Example 6.3. *The set of all polynomials in $\mathbb{C}[t]$ such that $\int_0^1 f dt = 0$, is a Mathieu subspace of $\mathbb{C}[t]$. In fact, if a polynomial f is such that $\int_0^1 f^m dt = 0$ for almost all m , then $f = 0$.*

Example 6.4. *The set of all polynomials f in $\mathbb{C}[t]$ such that $\int_0^\infty f e^{-t} dt = 0$, is a Mathieu space of $\mathbb{C}[t]$. In fact, if a polynomial f is such that $\int_0^\infty f^m e^{-t} dt = 0$ for almost all m , then $f = 0$.*

A beautiful proof of the result described in Example 6.3 was given by Mitya Boyarchenko in a personal communication to Zhao. This proof has been included in the recent preprint [FPYZ]. Using his techniques we were able to prove the statement described in Example 6.4 (see [EW rZ] and section 8 below). The importance of this result is due to the fact that it implies the one dimensional Image Conjecture $IC(1)$ (for a proof we refer to the next section). This implication reveals a remarkable fact, namely that the classical Laguerre polynomials play a special role in the study of the Image Conjecture and hence also of the Jacobian Conjecture.

To conclude this section let us observe that the apparently stronger statements in the second half of both Examples 6.3 and 6.4, are in fact equivalent to the ones made in the first halves of these examples. To see this, we need the following result of ([EZ2]).

Let B be an open set of \mathbb{R} and σ a positive measure on \mathbb{R} such that $\int_B f d\sigma$ is finite for all $f \in \mathbb{C}[t]$ and such that the \mathbb{C} -bilinear form defined by $\langle f, g \rangle = \int_B f g d\sigma$ is non-singular, i.e. for each non-zero f there exists a g such that $\langle f, g \rangle$ is non-zero.

Proposition 6.5. *If the set of all polynomials f with $\int_B f d\sigma$ is a Mathieu subspace of $\mathbb{C}[t]$, then the only polynomial f such that $\int_B f^m d\sigma = 0$ for almost all m , is the zero polynomial, i.e. $f = 0$.*

The announced equivalence of the statements made in both Examples 6.3 and 6.4 then follows by taking $B = (0, 1)$, $d\sigma = dt$ and $B = (0, \infty)$, $d\sigma = e^{-t} dt$ respectively: in both cases the hypothesis of the proposition is satisfied, since the corresponding bilinear forms are in fact Hermitian inner products on $\mathbb{C}[t]$.

7 The one dimensional Image Conjecture

In this section we show how the result of Example 6.4 implies $IC(1)$.

Theorem 7.1. $IC(1)$ is true.

Proof. Let L be the \mathbb{C} -linear map from $A = \mathbb{C}[\zeta, z]$ to $\mathbb{C}[z]$ defined in section 2 and 3 by the formula $L(\zeta^a z^b) = \partial^a(z^b)$. So this expression is zero if a is larger than b . Furthermore for any non-zero polynomial g we define its degree, denoted $\text{Deg}(g)$, as the maximum of the degrees of all non-zero monomials appearing in g , where $\text{Deg}(c\zeta^a z^b) = b - a$ and c is a non-zero constant in \mathbb{C} . It follows that

$$\text{if } \text{Deg}(g) \leq -1, \text{ then } L(g) = 0 \quad (3)$$

In particular, if for some element f of A its degree is ≤ -1 , then certainly the degrees of all powers f^m are ≤ -1 , which by (3) implies that $L(f^m) = 0$ for all $m \geq 1$. Now we will show that the converse is true as well.

Proposition 7.2. Let $f \in A$. Then $L(f^m) = 0$ for almost all m , if and only if $\text{Deg}(f) \leq -1$.

Before we prove this proposition let us show how it implies Theorem 7.1. So assume that $L(f^m) = 0$ for almost all m . Then by the proposition $\text{Deg}(f) \leq -1$, so the degree of f^m is at most $-m$. Now let g in A be non-zero and let d be its degree. It then follows that the degree of gf^m is at most -1 , if m is at least $d + 1$. The result then follows from (3).

To prove proposition 7.2, let f be such that $L(f^m) = 0$ for almost all m . Assume that $r := \text{Deg}(f) \geq 0$. We will arrive at a contradiction. Namely, let $g = \zeta^r f$. Then $\text{Deg}(g) = 0$. Furthermore

$$L(g^m) = L(\zeta^{mr} f^m) = \partial^{mr} L(f^m) = 0 \quad (4)$$

for almost all m . Writing g in its homogeneous decomposition, using that the degree of g is zero, we get that $g = g_0 + g_{-1} + \dots$ and hence that $g^m = g_0^m + g_*$, where g_0 is non-zero and g_* has degree at most -1 . Applying L to the last equality it follows from (3) and (4) that $L(g_0^m) = L(g^m) = 0$ for almost all m . Summarizing, if $r \geq 0$ there exists a non-zero element g_0 , which is homogeneous of degree zero, such that $L(g_0^m) = 0$ for almost all m .

Write again g instead of g_0 . Then g is a sum of monomials of the form $c_a \zeta^a z^a$. In other words $g = P(u)$, a non-zero polynomial in $u := \zeta z$ over \mathbb{C} . Now observe that $L(u^n) = L(\zeta^n z^n) = n!$. Since, as one easily verifies, $\int_0^\infty u^n e^{-u} du = n!$ for all $n \geq 0$, it follows that $L(f(u)) = \int_0^\infty f(u) e^{-u} du$ for each polynomial $f(u)$. Since $L(g^m) = L(P(u)^m) = 0$ for almost all m , it then follows from Example 6.4 that $P(u) = 0$, i.e. $g = 0$, a contradiction since $g = g_0$ is non-zero.

The proof given above shows that it is interesting and necessary to investigate $IC(n)$ for polynomials of the form $f(u_1, \dots, u_n)$, where $u_i = \zeta_i z_i$. One easily verifies that $L(u_1^{a_1} \dots u_n^{a_n}) = a_1! \dots a_n!$. This leads to the following question: if $L(f^m) = 0$ for almost all m , does this imply that $f = 0$? We don't know the answer to this question when n is at least two. However various computations suggest that the following is true: if f is a sum of N monomials and $L(f^m) = 0$ for the first N exponents m , then $f = 0$. Since, similar as in the proof of $IC(1)$ given above, one has the equality

$$L(f(u_1, \dots, u_n)) = \int_{(0, \infty)^n} f(u_1, \dots, u_n) e^{-(u_1 + \dots + u_n)} du_1 \dots du_n$$

we make the following conjecture

Conjecture 7.3. Let $\mathbb{C}[u] = \mathbb{C}[u_1, \dots, u_n]$ and $f \in \mathbb{C}[u]$.

Let $B = (0, \infty)^n$. If $\int_B f(u)^m e^{-(u_1 + \dots + u_n)} du = 0$ for almost all m , then $f = 0$.

Observe again, that this conjecture is a stronger version of a special case of Zhao's Integral conjecture. We refer the reader to the paper [EWZ], where several special cases of this conjecture are proved.

On the other hand, proving conjecture 7.3 is not enough to prove the Image Conjecture $IC(n)$ in higher dimension. Namely the following result of [Z3] shows, that there exist many polynomials f in $\mathbb{C}[\zeta, z]$, which do not belong to $\mathbb{C}[u_1, \dots, u_n]$, but do have the property that $L(f^m) = 0$ for all positive m .

Proposition 7.4. *Let $H = (H_1, \dots, H_n)$ in $\mathbb{C}[z]^n$ be such that each H_i has no terms of degree at most one. Let*

$$f(\zeta, z) = \zeta_1 H_1 + \dots + \zeta_n H_n.$$

Then $L(f^m) = 0$ for all positive m if and only if the Jacobian matrix of H is nilpotent.

8 Final remarks on the Image Conjecture.

To conclude this paper we will discuss a slightly stronger version of $IC(n)$, which we denote by $IC_*(n)$. At this moment it is not known if it is really stronger. First we introduce some notations.

Let \mathfrak{E} be a collection of fields. We make the following conjecture

$IC_*(\mathfrak{E}, n)$. *For every pair of positive integers d and e there exists a positive integer $D(d, e)$, such that the following holds: for any field k in \mathfrak{E} , any b in \mathbb{N}^n with $|b| = e$ and any polynomial f in ζ, z over k of degree d , such that $L(f^m) = 0$, for all positive m , we have that $L(z^b f^m) = 0$ for all $m \geq D(d, |b|)$.*

If \mathfrak{E} consists of only one field k , we simply write $IC_*(k, n)$ instead of $IC_*(\{k\}, n)$. Finally, denote the set of number fields by \mathfrak{N} .

Reduction theorem. *If $IC_*(\mathfrak{N}, n)$ holds, then $IC_*(\mathbb{C}, n)$ holds.*

Proof. i) First we show that the hypothesis implies that $IC_*(\overline{\mathbb{Q}}, n)$ holds, where $\overline{\mathbb{Q}}$ is the algebraic closure of \mathbb{Q} . Namely, let $f \in \overline{\mathbb{Q}}[\zeta, z]$, of degree d , be such that $L(f^m) = 0$, for all positive m . Since f has only a finite number of coefficients, which are all algebraic over \mathbb{Q} , it follows that all

these coefficients belong to some number field. Then the hypothesis implies that, for each monomial z^b , there exists a positive integer $D(d, |b|)$, such that $L(z^b f^m) = 0$ for all $m \geq D(d, |b|)$. So $IC_*(\overline{\mathbb{Q}}, n)$ holds.

ii) Now we will show that $IC_*(\mathbb{C}, n)$ holds. Let d be some positive integer and denote by f_U the universal polynomial of degree d in ζ and z , i.e. the coefficient of each monomial $\zeta^p z^q$, where $|p| + |q| \leq d$, is a new variable $C_{p,q}$. Denote by $\mathbb{Z}[C]$ the polynomial ring in these variables over \mathbb{Z} and let N denote the number of these variables. For each positive m we have that $L(f_U^m)$ is a polynomial in z , with coefficients in $\mathbb{Z}[C]$. Let I be the ideal in $\overline{\mathbb{Q}}[C]$ generated by all these coefficients, i.e. for all positive m . By i) we know that $IC_*(\overline{\mathbb{Q}}, n)$ holds. So for each monomial z^b there exists a positive integer $D(d, |b|)$ having the property described in i). For each $m \geq D(d, |b|)$ we get polynomials $L(z^b f_U^m)$ in z , with coefficients in $\mathbb{Z}[C]$. The ideal in $\overline{\mathbb{Q}}[C]$ generated by all these coefficients we denote by J_b . By i) it follows that, if $c \in \overline{\mathbb{Q}}^N$ is a zero of I and z^b is some monomial, then c is also a zero of J_b . Let g_1, \dots, g_s be generators of the ideal J_b . It then follows from the Nullstellensatz, that there exists some natural number r , such that each polynomial g_i^r belongs to I .

Now we can finish the proof. Namely, let f be a polynomial of degree d in ζ, z over \mathbb{C} , such that $L(f^m) = 0$, for all positive m . Fix some monomial z^b . Let $c_{p,q}$ be the coefficient of $\zeta^p z^q$ in f . Then the vector c in \mathbb{C}^N , whose components are formed by the $c_{p,q}$, is a zero of I and hence of the g_i (since each g_i^r belongs to I). So c is a zero of J_b , which means that $L(z^b f^m) = 0$, for all $m \geq D(d, |b|)$.

A proof of $IC_*(\mathbb{C}, 1)$

By the reduction theorem it suffices to prove $IC_*(k, 1)$ for number fields k . So let k be such a field. Let d be a natural number and z^b some monomial in the single variable z . Put $D(d, b) = b + 1$. Suppose now that $f \in k[\zeta, z]$ is such that $L(f^m) = 0$ for all positive m . Looking at the proof of proposition 7.2, we see that it suffices to prove the following result

Lemma 8.1. *Let k be a number field and g in $k[u]$ such that $L(g^m) = 0$ for almost all m . Then $g = 0$.*

Here L is the k -linear map from $k[u]$ to k defined by $L(u^i) = i!$, for all i .

Proof of lemma 8.1. Assume that g is non-zero. Then we may assume that $g = u^s + c_{s+1}u^{s+1} + \dots + c_d u^d$ for some c_i in k . Since k is a number field,

there exists, for almost all prime numbers p , a non-archimedean valuation v on k , such that $v(p) = 1$ and $v(c_i) \geq 0$ for all i . Now choose such a p large enough, with $L(g^p) = 0$. We claim that the equation $L(g^p) = 0$ leads to a contradiction, which shows that our assumption, that g is non-zero, is false. To obtain this contradiction, first observe that

$$g^p = u^{sp} + \sum_{i=s+1}^d c_i^p u^{ip} + p \sum_{i=sp+1}^{dp-1} h_i(c) u^i$$

where $h_i(c)$ belongs to the subring of k , generated by the c_i over \mathbb{Z} . In particular $v(h_i(c)) \geq 0$ for all i . Now applying L to the equality above, using that $L(g^p) = 0$ and $L(u^i) = i!$ for all i , gives

$$0 = (sp)! + \sum_{i=s+1}^d c_i^p (ip)! + p \sum_{i=sp+1}^{dp-1} h_i(c) i!$$

Observe that, if $i \geq s+1$, then $(ip)! = ip(sp)!n_i$, for some natural number n_i and that, if $i \geq sp+1$, then $i! = q_i(sp)!$, for some natural number q_i . Then, dividing the last equation by $(sp)!$, gives

$$1 + \sum_{i=s+1}^d c_i^p (ip)n_i + p \sum_{i=sp+1}^{dp-1} h_i(c) q_i = 0.$$

Finally observe that, each term in each of the two sums, has a positive valuation at v , since $v(p) = 1 > 0$ and both $v(c_i), v(h_i(c)) \geq 0$, for all i . Since $v(1) = 0$, this gives a contradiction.

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